

THE SPECTRAL DENSITY FUNCTION OF A TORIC VARIETY

D. BURNS, V. GUILLEMIN, AND A. URIBE

CONTENTS

1. Introduction	1
2. Toric varieties	4
3. Kähler reduction	6
4. Asymptotics	7
5. Riemann sums	11
6. The Shiffman–Tate–Zelditch results	12
7. Monomials and delta functions	14
References	16

ABSTRACT. For a Kähler manifold X, ω with a holomorphic line bundle L and metric h such that the Chern form of L is ω , the spectral measures are the measures $\mu_N = \sum |s_{N,i}|^2 \nu$, where $\{s_{N,i}\}_i$ is an L^2 -orthonormal basis for $H^0(X, L^{\otimes N})$, and ν is Liouville measure. We study the asymptotics in N of μ_N for X, L a *Hamiltonian* toric manifold, and give a very precise expansion in terms of powers $1/N^j$ and data on the moment polytope Δ of the Hamiltonian torus K acting on X . In addition, for a character k of K and the unique unit eigensection s_{Nk} for the character Nk of the torus action on $H^0(X, L^{\otimes N})$, we give a similar expansion for the measures $\mu_{Nk} = |s_{Nk}|^2 \nu$. A final remark shows that the eigenbasis $\{s_k, k \in \Delta \cap \mathbb{Z}^{\dim K}\}$ is a Bohr-Sommerfeld basis in the sense of [9], and that the asymptotic results of [1] are exact in this case.

Some of the present results are closely related to earlier results of [8]. The present paper uses no microlocal analysis, but rather an Euler-Maclaurin formula for Delzant polytopes.

1. INTRODUCTION

The purpose of this note is to explore a fundamental problem in spectral theory in the context of “toric geometry”. This problem, formulated in the context of Riemannian geometry, is the following: Let M be a compact Riemannian manifold, and let $\varphi_i, i = 1, 2, \dots$ be an orthonormal basis of eigenfunctions of the Laplace operator. What can one say about the spectral measures

$$(1.1) \quad \mu_i = |\varphi_i|^2 dx$$

as i tends to infinity? For instance, if the geodesic flow on T^*M is ergodic, Schnirelman-Colin de Verdière-Zelditch proved that along “most” subsequences $i_1, i_2, \dots, \mu_{i_j}$ tends weakly to the volume measure, dx . (This phenomenon is known as “quantum ergodicity”, and its violation by certain exceptional sequences of φ_i ’s

D.B. supported in part by NSF grant DMS-0514070.

V.G. supported in part by NSF grant DMS-0408993.

A.U. supported in part by NSF grant DMS-0401064.

as “quantum scarring”.) However, what can one say about the limiting behavior of the μ_i ’s if one makes other assumptions about geodesic flow, e. g. that it be periodic or completely integrable? If geodesic flow is periodic then it is known that for each geodesic γ there exists a sequence of “quasi-modes”, ϕ_i , $i = i_1, i_2, \dots$, such that μ_i tends in the limit to a delta function on γ . On the other hand it is also known that the eigenvalues of $\sqrt{\Delta}$ clump into clusters,

$$\lambda_{i,k}, \quad k = 1, \dots, N_i$$

with $|\lambda_{i,k} - (ai + b)| = O(i^{-1})$ (for suitable constants a, b), and it is known that a vestige of quantum ergodicity survives: The measures

$$(1.2) \quad \nu_i = \sum_k \mu_{i,k} = \sum_k |\varphi_{i,k}|^2 dx$$

tend in limit to the volume measure. (The simplest example is S^{n-1} . In this case ν_i is $\text{SO}(n)$ invariant and hence is the volume measure up to a constant factor.)

There is another important instance in which the eigenvalues can be clumped into clusters: If a compact Lie group K acts on M by isometries one can decompose the eigenspaces of Δ into K -invariant subspaces, and consider the spectral measures (1.2), where the $\varphi_{i,k}$ ’s are orthonormal bases of these subspaces. If K is an n -torus it is also natural to study the asymptotic behavior of the measure (1.1) not for arbitrary sequences of φ_i s, but for sequences for which φ_i lies in a weight space of K of weight α_i and the α_i tend asymptotically to infinity along a ray in \mathfrak{k}^* . In both these cases one would like to be able to relate the asymptotics of μ_i and ν_i to properties of the geodesic flow.

These problems have analogues in Kähler geometry: If X is a compact Kähler manifold and $L \rightarrow X$ a Hermitian line bundle whose curvature form is the negative of the Kähler form, then one can consider the asymptotic behavior of the measures

$$\mu_N = \sum |\varphi_{N,k}|^2 \nu$$

where $\{\varphi_{N,k} ; k = 1, \dots, d_N\}$ is an orthonormal basis of $\Gamma_{\text{hol}}(L^N)$ (holomorphic sections of (L^N)) and ν is Liouville measure.¹ (Notice that μ_N is now a measure on “phase space”. The analogue of X in the case of periodic geodesic flow is the quotient of the unit cotangent bundle of M by the flow.) Using general results about the microlocal structure of Szegő kernels, [2], one can prove that the μ_N have a weak asymptotic expansion as $N \rightarrow \infty$ with leading term Liouville measure.

If there is an action on X of a torus, K , preserving the Kähler structure and preserving L , one can decompose the spaces $\Gamma_{\text{hol}}(L^N)$ into weight spaces and, as above, study the asymptotics of the measure $|\varphi_{N,k}|^2 \nu$ associated with sequences of weights which tend asymptotically to infinity along rays in \mathfrak{k}^* .

The purpose of this article is to examine both of these problems in the setting of “toric geometry”. As a toric variety (together with its canonical Kähler metric) is completely determined by its moment polytope, it is natural to seek results formulated explicitly in terms of the polytope.

¹This measure can be defined intrinsically as the measure

$$C^\infty(X) \ni f \mapsto \text{Trace } \Pi_N M_f \Pi_N,$$

where Π_N is the orthogonal projection of $\Gamma(L^N)$ onto $\Gamma_{\text{hol}}(L^N)$, and M_f is the operator “multiplication by f ”.

In more detail, let K be an n -dimensional torus, X a (non-singular) K -toric variety, $\Phi : X \rightarrow k^*$ the moment map associated with the action of K on X and $\Delta = \Phi(X)$ the moment polytope. Under the action of K the space, $\Gamma_{\text{hol}}(L^N)$, breaks up into an orthogonal direct sum of one-dimensional weight spaces

$$\Gamma_{\text{hol}} = \bigoplus_{k \in [N\Delta]} \Gamma_k$$

indexed by the set $[N\Delta]$ of integer lattice points in the dilated polytope $N\Delta$, and thus

$$(1.3) \quad \mu_N = \sum \langle s_k, s_k \rangle \nu,$$

where $\{s_k \in \Gamma_k ; k \in [N\Delta]\}$ is an orthonormal basis of Γ_{hol} and $\langle s_k, s_k \rangle(p)$ is the norm-squared of $s_k(p) \in L_p^N$. Thus to understand the asymptotic behavior of μ_N one has to understand the asymptotic behavior of the functions $\langle s_k, s_k \rangle$. Our first step in this direction is the following explicit formula for this function. Let d be the number of facets of the polytope Δ , and $\ell_i : \Delta \rightarrow \mathbb{R}$, $i = 1, \dots, d$, the lattice distance to the i^{th} facet (see definition 2.1). Then

$$(1.4) \quad \langle s_k, s_k \rangle = \frac{1}{c_k} \left(\phi^* \exp \left(N \sum_{i=1}^d \ell_i \left(\frac{k}{N} \right) \log \ell_i - \ell_i \right) \right)$$

where c_k is the integral of the expression in parentheses.

The measure, μ_N , is K -invariant, so it is completely determined by its push-forward to X/K . Moreover, Φ is also K -invariant, so it defines a map $X/K \rightarrow \Delta$ which for toric varieties is a bijection. Hence to study the asymptotics of μ_N it suffices to study the asymptotics of the measure

$$\mu_N^\# =: \Phi_* \mu_N.$$

Moreover, for toric varieties $\Phi_* \nu$ is just ordinary Lebesgue measure on Δ , hence by (1.4) $\mu_N^\#$ is the measure

$$(1.5) \quad \mu_N^\# = \sum_{k \in [N\Delta]} \frac{1}{c_k} \exp \left(N \sum_{i=1}^d \ell_i \left(\frac{k}{N} \right) \log \ell_i - \ell_i \right) dx.$$

For x and y in Δ and $N \in \mathbb{Z}_+$ let

$$(1.6) \quad K_N(x, y) = c_N(x)^{-1} \exp \left(N \sum_{i=1}^d \ell_i(x) \log \ell_i(y) - \ell_i(y) \right)$$

where

$$(1.7) \quad c_N(x) = \int_{\Delta} \exp \left(N \sum_{i=1}^d \ell_i(x) \log \ell_i(y) - \ell_i(y) \right) dy.$$

Then, for $f \in C^\infty(\Delta)$,

$$(1.8) \quad \int_{\Delta} f d\mu_N^\# = \sum_{k \in [N\Delta]} f_N^\# \left(\frac{k}{N} \right)$$

where

$$(1.9) \quad f_N^\sharp(x) = \int_{\Delta} K_N(x, y) f(y) dy.$$

One of the main result of this paper is the following.

Theorem 1.1. *There exist, for $i = 0, 1, 2, \dots$, differential operators, $P_i(x, D) : \mathcal{C}^\infty(\Delta) \rightarrow \mathcal{C}^\infty(\Delta)$, of order $2i$ with the property*

$$(1.10) \quad \int_{\Delta} K_N(x, y) f(y) dy \sim \sum_{i=0}^{\infty} P_i(x, D) f N^{-i}.$$

Moreover, $P_0 = I$.

The P_i 's are *combinatorial* invariants of the polytope Δ (albeit given by rather complicated formulas). By combining this result with an Euler–Maclaurin formula for Riemann sums over polytopes (see [7]), we will be able to write the sum (1.5) as an asymptotic series in inverse powers of N in which the individual terms are integrals over the faces of Δ of differential expressions in f . Moreover, if k is a lattice point of Δ and $\mu_{N,k}$ is the measure

$$\mu_{Nk} = \langle s_{Nk}, s_{Nk} \rangle \nu$$

the formula (1.10) yields as a corollary a second main result of this paper:

Theorem 1.2. *For $f \in C^\infty(\Delta)$ one has an asymptotic expansion*

$$\int_X \phi^* f d\mu_{Nk} \sim \left(\sum_{i=0}^{\infty} P_i(x, D) f N^{-i} \right) |_{x=k}$$

where the P_i are the same operators as before.

In case k is in the interior of Δ this result follows from the results in §7 and the “matrix coefficients” estimates in [1]. However, we will give below a direct proof that includes the case $k \in \partial\Delta$.

To summarize briefly the contents of this article: In §2 we will review basic facts about toric varieties, in §3 derive the formula (1.4), in §4 prove Theorem 1.1 and in §5 derive from it the asymptotic expansion mentioned above. The asymptotic properties of s_k that we discuss in §4 are closely related to some results of Shiffman–Tate–Zelditch, and can be viewed as an alternative derivation of these results. (See [8]). We will comment on the relation of our work to theirs in §6. Also, on the open set where $\langle s_k, s_k \rangle$ is non-zero, $-\log \langle s_k, s_k \rangle$ is a potential for the Kähler metric on X , so inter alia our results give a formula for this Kähler potential in terms of moment polytope data. (For other formulas of this type see [3], [4] and [6].) Finally, in §7 we show that the basis of sections $\{s_k\}$ is a “Bohr–Sommerfeld basis” in the sense of Tyurin, [9].

It may be worth noting that we make no use of microlocal analysis in this paper.

2. TORIC VARIETIES

Let T be the standard d -dimensional torus, $T = (S^1)^d$, let $\mathfrak{t} = \text{Lie } T = \mathbb{R}^d$ and let e_1, \dots, e_d be the standard basis vectors of \mathbb{R}^d . T acts on \mathbb{C}^d by its diagonal action, and if we equip \mathbb{C}^d with the Kähler form, $\omega = \sqrt{-1} \sum dz_i \wedge d\bar{z}_i$ this becomes a Hamiltonian action with moment map

$$(2.1) \quad \phi : \mathbb{C}^d \rightarrow \mathfrak{t}^*, \quad z \rightarrow \sum |z_i|^2 e_i^*.$$

For G a codimension n subtorus of T , let $\mathfrak{g} = \text{Lie } G$, let $\mathbb{Z}_G^* \subset \mathfrak{g}^*$ be the weight lattice of G and let

$$(2.2) \quad L : \mathfrak{t}^* \rightarrow \mathfrak{g}^*$$

be the transpose of the inclusion map, $\mathfrak{g} \rightarrow \mathfrak{t}$. Then the action of T on \mathbb{C}^d restricts to a Hamiltonian action of G on \mathbb{C}^d with moment map

$$(2.3) \quad \psi = L \circ \phi = \sum |z_i|^2 \alpha_i$$

where $\alpha_i = Le_i^* \in \mathbb{Z}_G^*$. We will assume that this moment map is proper, or alternatively, that the α_i 's are "polarized": for some $\xi \in \mathfrak{g}$, all the numbers, $\alpha_i(\xi)$ are positive. The toric varieties we will be considering in this paper are symplectic reduced spaces of the form

$$(2.4) \quad X_\alpha = Z_\alpha / G$$

where α is in \mathbb{Z}_G^* and $Z_\alpha = \psi^{-1}(\alpha)$. We recall that the action of G on $\psi^{-1}(\alpha)$ is locally free iff α is a regular value of ψ , and since we will only be considering non-singular toric varieties in this paper we will assume that G acts freely on Z_α . Hence X_α is a manifold and the projection

$$(2.5) \quad \pi : Z_\alpha \rightarrow X_\alpha$$

is a principal G -fibration. From the action of T on \mathbb{C}^d we get a Hamiltonian action of T on X_α , and if we denote by " ι " the inclusion of Z_α into \mathbb{C}^d , the moment map for the T -action on \mathbb{C}^d is related to the moment map for the T action on X_α by the identity:

$$(2.6) \quad \phi_\alpha \circ \pi = \phi \circ \iota.$$

Thus we have a commutative diagram:

$$\begin{array}{ccccc} Z_\alpha & \xhookrightarrow{\iota} & \mathbb{C}^d & & \\ \pi \downarrow & & \phi \downarrow & \searrow \psi & \\ X_\alpha & \xrightarrow{\phi_\alpha} & \mathfrak{t}^* & \xrightarrow{L} & \mathfrak{g}^* \end{array}$$

The moment polytope for the action of T on X_α is

$$(2.7) \quad \Delta_\alpha = \mathbb{R}^d \cap L^{-1}(\alpha),$$

by (2.1) and (2.3). (Here we've identified \mathfrak{t}^* with \mathbb{R}^d via the basis vectors, e_i^* .) The facets of this polytope are the intersections of $L^{-1}(\alpha)$ with the coordinate hyperplanes $x_i = 0$ in \mathbb{R}^d .

Definition 2.1. The "lattice distance" to the i^{th} facet, ℓ_i , is the restriction of the coordinate function x_i to Δ_α .

Since G acts trivially on X_α the action of T on X_α is effectively an action of the quotient group, $K = T/G$, and since $\text{Lie } K = \mathfrak{k} = \mathfrak{t}/\mathfrak{g}$, the dual \mathfrak{k}^* is the annihilator in \mathfrak{t}^* of \mathfrak{g} , and hence is the kernel of the map L . To make the action of K a Hamiltonian action one has to normalize the moment map, $\phi_\alpha : X_\alpha \rightarrow L^{-1}(\alpha)$, so that it maps into \mathfrak{k}^* , and this one can do by fixing an element, $c_\alpha \in \mathbb{Z}^d \cap \text{Int } \Delta_\alpha$ and replacing ϕ_α by $\phi_\alpha - c_\alpha$. We won't, however, bother to make this normalization here and will continue to think of ϕ_α as a map into $L^{-1}(\alpha)$.

3. KÄHLER REDUCTION

In this section we review some general facts about Kähler reduction and use them to derive the formula (1.4). Let M be a complex manifold and $L \rightarrow M$ a holomorphic line bundle. We recall that if L is equipped with a Hamiltonian inner product, there is a unique holomorphic connection on L which is compatible with this inner product. More explicitly, if ∇ is a holomorphic connection and $\langle \cdot, \cdot \rangle$ an inner product then for every holomorphic trivialization $s : U \rightarrow L$

$$(3.1) \quad \frac{\nabla s}{s} = \mu \in \Omega^{1,0}(U)$$

and the compatibility of $\langle \cdot, \cdot \rangle$ and ∇ reduces to

$$(3.2) \quad d \log \langle s, s \rangle = \mu + \bar{\mu}$$

and hence

$$(3.3) \quad \partial \log \langle s, s \rangle = \mu$$

which shows that the inner product determines the connection and vice versa. It also shows that

$$(3.4) \quad \text{curv}(\nabla) = \sqrt{-1} \partial \bar{\partial} \log \langle s, s \rangle =: -\omega.$$

Suppose now that the form, ω , is Kähler. Let G be an m -dimensional torus, let $\tau : G \times M \rightarrow M$ be a holomorphic action of G on M and let $\tau^\sharp : G \times L \rightarrow L$ be an action of G on L by holomorphic line bundle automorphisms which is compatible with τ . If τ^\sharp preserves $\langle \cdot, \cdot \rangle$ then by (3.2)–(3.3) it preserves ∇ and ω . Moreover, by Kostant's formula there is an intrinsically defined moment map, $\Phi : M \rightarrow \mathfrak{g}^*$, such that

$$(3.5) \quad L_v s = \nabla_{v_M} s + i \langle \Phi, v \rangle s$$

for all $s \in \mathcal{C}^\infty(L)$ and $v \in \mathfrak{g}$. In other words the infinitesimal action of G on $\mathcal{C}^\infty(L)$ is completely determined by Φ and $\langle \cdot, \cdot \rangle$.

Let's now describe what symplectic reduction looks like from this Kählerian perspective. Given $\alpha \in \mathbb{Z}_G^*$, let $Z_\alpha = \Phi^{-1}(\alpha)$. Assuming that G acts freely on Z_α , the reduced space, $X_\alpha = Z_\alpha/G$ is a \mathcal{C}^∞ manifold, and the projection, $\pi : Z_\alpha \rightarrow X_\alpha$ is a principal G -fibration. Let $L_\alpha \rightarrow X_\alpha$ be the line bundle whose fiber at p is the (one-dimensional) space of sections

$$s : \pi^{-1}(p) \rightarrow L$$

which transforms under G by the recipe

$$(3.6) \quad \tau^\sharp(\exp v)^* s = e^{i\alpha(v)} s$$

or alternatively, by (3.5), are auto-parallel along $\pi^{-1}(p)$. For such a section, $\langle s, s \rangle$ is constant along $\pi^{-1}(p)$, so the inner product, $\langle \cdot, \cdot \rangle$, induces an inner product, $\langle \cdot, \cdot \rangle_\alpha$ on L_α . In terms of sections, if $\mathcal{C}^\infty(L)^\alpha$ is the space of global sections of L which transform by (3.6) and $\iota : Z_\alpha \rightarrow M$ is inclusion

$$(3.7) \quad \iota^* \mathcal{C}^\infty(L)^\alpha = \pi^* \mathcal{C}^\infty(L_\alpha)$$

and if $\iota^* s = \pi^* s_\alpha$

$$(3.8) \quad \iota^* \langle s, s \rangle = \pi^* \langle s_\alpha, s_\alpha \rangle_\alpha.$$

We now define a complex structure on X_α and make $L_\alpha \rightarrow X_\alpha$ into a holomorphic line bundle by requiring that

$$(3.9) \quad \pi^* \mathcal{O}_\alpha \subseteq \iota^* \mathcal{O}$$

and

$$(3.10) \quad \pi^* \mathcal{L}_\alpha \subseteq \iota^* \mathcal{L}$$

where \mathcal{O} is the sheaf of holomorphic functions on M , \mathcal{O}_α the sheaf of holomorphic functions on X , \mathcal{L} the sheaf of holomorphic sections of L and \mathcal{L}_α the sheaf of holomorphic sections of L_α . By (3.3) one gets a holomorphic connection, ∇_α , on L_α which is compatible with \langle, \rangle_α and by (3.4) and (3.8)

$$(3.11) \quad \iota^* \text{curv}(\nabla) = \pi^* \text{curv}(\nabla_\alpha).$$

Thus $-\text{curv}(\nabla_\alpha)$ is the reduced symplectic form on X_α .

We'll conclude this section by applying these general observations to the set-up in §2.

Let $L = \mathbb{C} \times \mathbb{C}^d \rightarrow \mathbb{C}^d$ be the trivial line bundle over \mathbb{C}^d and $s : \mathbb{C}^d \rightarrow L$ the trivial section, $s(z) = (z, 1)$. If we equip L with the Hermitian inner product, $\langle s, s \rangle = e^{-|z|^2}$, we get a non-trivial connection on L , and by (3.4) the curvature form of this connection is minus the symplectic form, $\sqrt{-1} \sum dz_i \wedge d\bar{z}_i$. If we let G act on L by requiring that s be G -invariant then the space $\Gamma_{\text{hol}}(L)^\alpha$ is spanned by monomials, $z_1^{k_1} \dots z_d^{k_d}$, for which $\alpha = L(\sum k_i e_i^*) = \sum k_i \alpha_i$, i.e., for which $k \in \mathbb{Z}^d \cap \Delta_\alpha$. For each of these sections let s_k be the corresponding holomorphic section of L_α . On Z_α one has

$$\langle z^k, z^k \rangle = |z_1|^{2k_1} \dots |z_d|^{2k_d} e^{-|z|^2},$$

and so by (3.8) and (2.1)

$$\pi^* \langle s_k, s_k \rangle_\alpha = \iota^* |z_1|^{2k_1} \dots |z_d|^{2k_d} e^{-|z|^2} = \iota^* \phi^*(x_1^{k_1} \dots x_d^{k_d} e^{-\sum x_i}).$$

But, by (2.6)

$$\iota^* \phi^*(x_1^{k_1} \dots x_d^{k_d} e^{-\sum x_i}) = \pi^* \phi_\alpha^*(\ell_1^{k_1} \dots \ell_d^{k_d} e^{-\sum \ell_i}).$$

Hence we conclude that

$$(3.12) \quad \langle s_k, s_k \rangle_\alpha = \phi_\alpha^*(\ell_1^{k_1} \dots \ell_d^{k_d} e^{-\sum \ell_i}),$$

which implies the formula (1.4).

4. ASYMPTOTICS

In this section we will use stationary phase to analyze the behavior of the integral (1.9) as N tends to infinity. Let

$$(4.1) \quad \varphi(x, y) = \sum \ell_i(x) \log \ell_i(y) - \ell_i(y)$$

be the phase function in this integral. We claim:

Lemma 4.1. *For x a fixed point in the interior of Δ , the function φ , regarded as a function of y , has a unique critical point at $x = y$, and this critical point is the unique global maximum of the function, φ , on Δ .*

Proof. Since

$$d\varphi = \sum \ell_i(x) \frac{d\ell_i}{\ell_i(y)} - d\ell_i = 0$$

at $x = y$, the point $x = y$ is a critical point of φ and since

$$(4.2) \quad d^2\varphi = - \sum \ell_i(x) \frac{(d\ell_i)^2}{\ell_i(y)^2}$$

this critical point is a maximum. Moreover, by (4.2) every critical point of φ in the interior of Δ has to be a maximum and as y tends to the boundary of Δ , φ tends to $-\infty$. Hence by the “peaks-passes” lemma x is the only critical point of φ and is its unique global maximum. \square

For x in the interior of a boundary face, F , of Δ , one has an analogous result:

Lemma 4.2. *The restriction of φ to F has a unique critical point at $x = y$, and this critical point is the unique global maximum of φ on Δ . In addition, the derivatives of φ (as a function of y) in directions normal to F are not zero at $y = x$.*

Proof. Suppose that F is defined by the equations $\ell_i = 0$, $i \in I \subset \{1, 2, \dots, d\}$. Then, for x in the interior of F ,

$$(4.3) \quad \varphi(x, y) = \sum_{i \notin I} \ell_i(x) \log \ell_i(y) - \sum_{i=1}^d \ell_i(y),$$

which shows that φ is a decreasing function of the ℓ_i , $i \in I$. The joint minimum of those functions is exactly F , and therefore the global maximum of $\Delta \ni y \mapsto \varphi(x, y)$ is attained on F . The restriction of this function to F is

$$\varphi(x, y)|_{y \in F} = \sum_{i \notin I} \ell_i(x) \log \ell_i(y) - \sum_{i \notin I} \ell_i.$$

Lemma 4.1 can be applied to this restriction, and therefore $y = x$ is the unique global maximum of $\varphi|_F$, and so x is the unique global maximum of $\Delta \ni y \mapsto \varphi(x, y)$ on Δ . Moreover, at $y = x$

$$(4.4) \quad (d\varphi)_y = - \sum_{i \in I} (d\ell_i)_y,$$

which shows that the derivatives normal to F are not zero at x (and φ decreases to the interior of the polytope). \square

From these lemmas one obtains the following “localization” theorem for the integral operator defined by (1.10).

Theorem 4.3. *Let f and g be in $C^\infty(\Delta)$. Suppose that for $x \in \Delta$, $f(x) \neq 0$ and $x \notin \text{supp } g$. Then*

$$(4.5) \quad \left| \int_{\Delta} e^{N\varphi(x,y)} g(y) dy \right| \leq e^{-cN} \left| \int_{\Delta} e^{N\varphi(x,y)} f(y) dy \right|$$

for some positive constant, c .

We will now examine the local behavior of the transform (1.9) in the neighborhood of a fixed vertex, p , of Δ . Let Δ_p be the open subset of Δ obtained by deleting from Δ all facets except the facets containing p . By repagination we can assume that these are the facets, $\ell_i = 0$, $i = 1, \dots, n$.

Lemma 4.4. *There exists an affine transform mapping Δ_p onto an open subset of the positive orthant, \mathbb{R}_+^n , mapping p onto the origin and transforming the ℓ_i 's, $i = 1, \dots, n$, into the coordinate functions, x_i , $i = 1, \dots, n$.*

(For proof of this “standard fact” about moment polytopes of toric manifolds see [5].)

In these new coordinates the phase function (4.1) takes the form

$$(4.6) \quad \varphi(x, y) = \sum x_i \log y_i - y_i + \psi(x, y)$$

where

$$(4.7) \quad \psi(x, y) = \sum_{r>n} \ell_r(x) \log \ell_r(y) - \ell_r(y)$$

is a \mathcal{C}^∞ function on Δ_p . Moreover, with x fixed, the derivative of ψ with respect to y :

$$d\psi = \sum_{r>n} \frac{\ell_r(x)}{\ell_r(y)} d\ell_r - d\ell_r$$

is zero at $x = y$, so

$$(4.8) \quad \frac{\partial \psi}{\partial y_i}(x, y) = \sum h_{i,j}(x, y)(x_j - y_j)$$

and

$$\begin{aligned} \frac{\partial \varphi}{\partial y_i} &= \frac{x_i - y_i}{y_i} + \sum_j h_{i,j}(x, y)(x_j - y_j) \\ &= \frac{1}{y_i} \sum_j (\delta_{i,j} + y_i h_{i,j})(x_j - y_j). \end{aligned}$$

Hence

$$(4.9) \quad x_j - y_j = \sum g_{i,j}(x, y) y_i \frac{\partial \varphi}{\partial y_i}$$

the $g_{i,j}$'s being \mathcal{C}^∞ in a neighborhood of $x = y = 0$.

Consider now an integral of the form

$$(4.10) \quad \int_{\mathbb{R}_+^n} e^{N\varphi(x, y)} f(x, y) dy$$

where f is \mathcal{C}^∞ and supported in a neighborhood of $x = y = 0$. Let $\rho(y)$ be a \mathcal{C}_0^∞ function which is equal to one on a neighborhood of the support of f . Then

$$(4.11) \quad f(x, y) = \left(f_0(x) + \sum (y_j - x_j) f_j^\sharp(x, y) \right) \rho(y),$$

where

$$(4.12) \quad f_0(x) = f(x, x)$$

and

$$(4.13) \quad f_j^\sharp(x, y) = \int_0^1 \frac{\partial}{\partial y_j} f(x, x + t(y - x)) dt.$$

By (4.9) and (4.11) we can write (4.10) as the sum of the two expressions

$$(4.14) \quad f_0(x) \int_{\mathbb{R}_+^n} e^{N\varphi(x,y)} \rho(y) dy$$

and

$$(4.15) \quad - \int_{\mathbb{R}_+^n} e^{N\varphi(x,y)} \left(\sum_{i,j} g_{i,j} f_j^\# y_i \frac{\partial \varphi}{\partial y_i} \right) \rho(y) dy$$

and by making the substitution

$$e^{N\varphi(x,y)} \frac{\partial \varphi}{\partial y_i} = \frac{1}{N} \frac{\partial}{\partial y_i} e^{N\varphi(x,y)}$$

and integrating by parts with respect to y_i we can rewrite (4.15) in the form

$$(4.16) \quad \frac{1}{N} \int_{\mathbb{R}_+^n} e^{N\varphi(x,y)} f_1(x, y) dy$$

where

$$(4.17) \quad f_1(x, y) = \sum_{i,j} \frac{\partial}{\partial y_i} (y_i f_j^\# g_{i,j} \rho(y)).$$

(Notice that in integrating by parts we don't pick up boundary terms because of the presence of the y_i 's in the integrand.)

From (4.14) and (4.16) we get for (4.10) the expansion

$$(4.18) \quad f_0(x) \int_{\mathbb{R}_+^n} e^{N\varphi(x,y)} \rho(y) dy + \frac{1}{N} \int_{\mathbb{R}} e^{N\varphi(x,y)} f_1(x, y) dy$$

and by iteration of (4.18), an expansion

$$(4.19) \quad \sum_{i=0}^{k-1} f_i(x) N^{-i} \int_{\mathbb{R}_+^n} e^{N\varphi(x,y)} \rho(y) dy + R_k(x)$$

where

$$(4.20) \quad R_k(x) = N^{-k} \int_{\mathbb{R}_+^n} e^{N\varphi(x,y)} f_k(x, y) dy.$$

(In more detail: $f_i(x) = f_i(x, x)$ and $f_i(x, y)$ is obtained from $f(x, y)$ by iterating i times the operation (4.17). In particular f_i is a sum of derivatives of f of degree less than or equal to $2i$ with \mathcal{C}^∞ functions as coefficients.)

Finally observe that for x near zero the quotient of

$$\int_{\Delta} e^{N\varphi(xy)} (1 - \rho(y)) dy$$

by

$$(4.21) \quad \int_{\Delta} e^{N\varphi(x,y)} dy$$

is of order $O(e^{-cN})$ by Theorem 4.3, hence if we divide the sum (4.19) by (4.21) and let k tend to infinity we get the asymptotic expansion (1.10).

Remark 4.5. In the discussion above we've assumed that $f(x, y)$ is supported on the set, $x, y \in \Delta_p$, however, by the localization Theorem 4.3 one can always reduce to this case by means of a partition of unity.

5. RIEMANN SUMS

One of the many variants of the classical Euler–Maclaurin formula asserts that for $f \in \mathcal{C}_0^\infty(\mathbb{R})$ the Riemann sum

$$\frac{1}{N} \sum_{k=0}^{\infty} f\left(-\frac{k}{N}\right)$$

differs from the Riemann integral

$$\int_{-\infty}^0 f(x) dx$$

by an asymptotic series

$$(5.1) \quad \frac{f(0)}{2N} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n}{(2n)!} f^{(2n-1)}(0) N^{-2n}$$

where the B_n 's are the Bernoulli numbers.

Recalling that

$$\tau(s) =: \frac{s}{1 - e^{-s}} = 1 + \frac{s}{2} + \sum (-1)^{n-1} B_n \frac{s^{2n}}{(2n)!}$$

this asymptotic expansion can be written more succinctly in the form:

$$(5.2) \quad \frac{1}{N} \sum_{k=0}^{\infty} f\left(-\frac{k}{N}\right) \sim \left(\tau\left(\frac{1}{N} \frac{\partial}{\partial h}\right) \int_{-\infty}^h f(x) dx \right) (h=0).$$

Guillemin and Sternberg have recently announced in [7] an n -dimensional version of this result in which the interval, $(-\infty, 0]$, gets replaced by a convex polytope. In particular for the moment polytopes associated with toric manifolds their formula is basically a “product” version of the formula above and is proved by localization arguments similar to those we used above to prove Theorem 1.1. Let $\Delta \subseteq \mathbb{R}^n$ be such a polytope and let d be the number of facets of Δ . Then Δ can be defined by a set of inequalities

$$(5.3) \quad \langle u_i, x \rangle \leq c_i$$

where c_i is an integer and $u_i \in (\mathbb{Z}^n)^*$ is a primitive lattice vector which is perpendicular to the i^{th} facet and points “outward” from Δ . The Euler–Maclaurin formula in [7] asserts:

Theorem 5.1. *Let Δ_h be the polytope*

$$(5.4) \quad \langle u_i, x \rangle \leq c_i + h_i, \quad i = 1, \dots, d.$$

Then for $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$

$$(5.5) \quad \frac{1}{N^n} \sum_{k \in \mathbb{Z}^n \cap N\Delta} f\left(\frac{k}{N}\right) \sim \left(\tau\left(\frac{1}{N} \frac{\partial}{\partial h}\right) \int_{\Delta_h} f(x) dx \right) (h=0)$$

where $\tau(s_1, \dots, s_d) = \tau(s_1) \dots \tau(s_d)$.

Now notice that if we divide (1.8) by N^n the right hand side is exactly a Riemann sum of the form above. Hence if we plug in for f_N^\sharp the asymptotic expansion (1.10) and apply to each summand the formula (5.5) we obtain an “Euler–Maclaurin formula” for the asymptotics of the measure μ_N .

6. THE SHIFFMAN–TATE–ZELDITCH RESULTS

Let $\varphi(x, y)$ be the function (4.1). By (1.4)

$$(6.1) \quad \langle s_k, s_k \rangle(p) = \frac{1}{c_k(x)} e^{N\varphi(x, y)}$$

where $x = k/N$, $p \in \Phi^{-1}(y)$ and

$$(6.2) \quad c_k(x) = \int_{\Delta} e^{N\varphi(x, y)} dy.$$

If $x \in \text{Int } \Delta$ then by Lemma 4.1 the function

$$\Delta \ni y \mapsto \varphi(x, y)$$

has a unique non-degenerate maximum at $y = x$, and hence by the lemma of steepest descent

$$(6.3) \quad c_k(x) = \left(\frac{2\pi}{N} \right)^{n/2} h(x)^{-\frac{1}{2}} e^{N\varphi(x, x)} (1 + o(N^{-1}))$$

where $h(x)$ is the determinant of the quadratic form

$$(6.4) \quad \sum \frac{1}{\ell_i(x)} (d\ell_i)^2(x).$$

Thus as $k = Nx$ tends to infinity along the ray through x one gets the asymptotic identity

$$(6.5) \quad \langle s_k, s_k \rangle(p) \sim \left(\frac{N}{2\pi} \right)^{n/2} h(x)^{1/2} e^{N(\varphi(x, y) - \varphi(x, x))}$$

at $p \in \Phi^{-1}(y)$. In particular, as N tends to infinity $\langle s_k, s_k \rangle$ concentrates exponentially on the Bohr–Sommerfeld set, $\Phi^{-1}(k/N)$. This result is due to Shiffman, Tate and Zelditch,² who also observe that by applying steepest descent arguments to the function (4.3) one gets an analogue of (6.5) for x lying in the interior of a face, F , of Δ . In this case the asymptotic dependence of $\langle s_k, s_k \rangle(p)$ on N is given by an expression similar to (6.5), except that the “ n ” in (6.5) has to be replaced by the dimension of F . Hence the behavior of $\langle s_k, s_k \rangle(p)$ for $k = Nx$ is very non-uniform in x when x is near the boundary of Δ . We will prove below that by averaging their result over an “ δ -pinched” neighborhood

$$\left| \frac{k}{N} - y \right| < \frac{1}{N\delta}, \quad 0 < \delta < \frac{1}{2}$$

one gets a version of (6.5) which is much more uniform in k/N :

²More or less: Their result involves a slightly different choice of inner product on the s_k ’s and of coordinates on X . See [8].

Theorem 6.1. *For $x = k/N \in \Delta$, $\delta \in (0, 1/2)$, and for every test function $\psi \in C_0^\infty(\mathbb{R}^n)$,*

$$(6.6) \quad \int_{\Delta} \langle s_k, s_k \rangle \psi \left(N^\delta \left(\frac{k}{N} - y \right) \right) dy \sim \sum_{i=0}^{\infty} \sigma_i(x) N^{-(1+2\delta)i},$$

the $\sigma_i(x)$ being C^∞ functions on Δ . Thus the averaged estimate, unlike the point-wise estimate (6.5), is “uniform up to the boundary”.

Proof. We mimic the integration by parts argument in §4. Applying this argument to the function

$$(6.7) \quad f(x, y) = \psi \left(N^\delta (x - y) \right)$$

and keeping track of powers of N one gets for the integral

$$\int_{\mathbb{R}_+^n} e^{N\varphi(x, y)} \psi \left(N^\delta (x - y) \right) dy$$

an expression

$$\sum_{i=0}^{k-1} \sigma_i(x) N^{-i(1-2\delta)} \int_{\mathbb{R}_+^n} e^{N\varphi(x, y)} \rho(y) dy + R_k(x)$$

where

$$R_k(x) = N^{-k(1-2\delta)} \int_{\mathbb{R}_+^n} e^{N\varphi(x, y)} f_k(x, y, N^\delta(x - y)) dy,$$

and dividing by (4.21) and letting k tend to infinity one gets the estimate (6.6). \square

Another result of [8] which is closely related to the results of this paper concerns the asymptotic behavior of another interesting measure associated with the norm-squares of the s_k ’s, namely the measure on the real line

$$(6.8) \quad \mu_N([t, \infty]) = \text{Vol} \{ \langle s_k, s_k \rangle(p) \geq t \}$$

(i. e. the distribution function of the “random variable” $\langle s_k, s_k \rangle$), where $k = Nx$. Assuming that x is a point in the interior of Δ , Shiffman, Tate and Zelditch prove that the moments of this measure have the limiting behavior

$$(6.9) \quad \int_0^\infty t^m d\mu_N \sim (c N^{n/2})^{m-1} m^{-n/2}, \quad m = 0, 1, \dots$$

where c is a constant depending on x , and from this result deduce that μ_N satisfies “universal rescaling laws” in various regimes (e. g. for t exponentially small with respect to N or for t greater than some positive power of N). To deduce (6.9) from the results above we note that the integral on the left is just

$$\int_X \langle s_k, s_k \rangle^m d\nu$$

where ν is, as in §1, Liouville measure. This integral is equal to the integral over Δ of the right-hand side of (6.1) to the m^{th} power, with respect to Lebesgue measure. Using (6.2), this gives

$$\int_X \langle s_k, s_k \rangle^m d\nu = \frac{c_{km}(x)}{c_k(x)^m}.$$

But by (6.3)

$$(6.10) \quad \frac{c_{km}(x)}{c_k(x)^m} \sim \left(\frac{N}{2\pi}\right)^{\frac{(m-1)n}{2}} m^{-n/2} h(x)^{\frac{m-1}{2}},$$

and we recover (6.9). (We are grateful to Zuoqin Wang for pointing out to us this connection between (6.1)–(6.3) and these rescaling laws of [8].)

7. MONOMIALS AND DELTA FUNCTIONS

In this section we present a precise way to relate the sections s_k with the Bohr-Sommerfeld fibers of the moment map. This can be seen as a concrete realization of the expected (or hoped-for) equivalence between the complex polarization used in this paper and the singular real polarization defined by such fibers. The result is exact (not asymptotic), so in this section $N = 1$.

Let $P \subset L^*$ be the unit circle bundle, which is a principal S^1 bundle with connection. We denote by $\mathcal{H} \subset L^2(P)$ the L^2 closure of the space of smooth functions that extend holomorphically to the unit disk bundle of L^* , and let $\Pi : L^2(P) \rightarrow \mathcal{H}$ be the orthogonal projection. Under the circle action \mathcal{H} splits into isotypical subspaces,

$$\mathcal{H} = \bigoplus_N \mathcal{H}_N.$$

Specifically, \mathcal{H}_N consists of eigenspace of the infinitesimal generator of the S^1 action in \mathcal{H} corresponding to the eigenvalue $\sqrt{-1}N$. \mathcal{H}_1 is naturally isomorphic with the space of holomorphic sections of L . If $s : X \rightarrow L$ is such a section, we will denote by

$$s^\flat \in \mathcal{H}_1$$

the corresponding function on P .

Since the torus K acts on the bundle $L \rightarrow X$ (preserving the hermitian structure) it acts on P , preserving the connection. The the infinitesimal action of K on P is given by the Kostant formula, (3.5), translated into this setting:

$$(7.1) \quad \forall A \in \mathfrak{k} \quad \xi_A^\# = \tilde{\xi}_A + H_A \partial_\theta.$$

Here:

- (1) ξ_A is the vector field on X induced by A ,
- (2) $\tilde{\xi}_A$ is the horizontal lift of ξ_A , and
- (3) H_A is the A -component of the moment map $X \rightarrow \mathfrak{k}^*$, pulled back to P .

Note that, since H_A is constant along trajectories of $\tilde{\xi}_A$, the two fields on the right-hand side of (7.1) commute. Furthermore, the representation of the torus K on $L^2(P)$ by translations commutes with the projection, Π . Therefore, if $A \in \mathfrak{k}$,

$$[\mathcal{L}_{\xi_A^\#}, \Pi] = 0,$$

where \mathcal{L} denotes the Lie derivative.

We begin with:

Lemma 7.1. *Let $k \in [\Delta]$ be a lattice point. Then there exists a closed submanifold $Y_k \subset P$ such that*

- (1) Y_k is horizontal, and the projection, $P \rightarrow X$, restricted to Y_k is a diffeomorphism onto $\phi^{-1}(k)$.
- (2) The restriction of s_k^\flat to Y_k is a non-zero constant function.

Proof. The inverse image $\phi^{-1}(k)$ is an orbit of K , and therefore diffeomorphic to a quotient torus, K/K_k , where K_k is the isotropy subgroup of any point in $\phi^{-1}(k)$. The Lie algebra of K_k is the conormal space to the face, F , of Δ such that $k \in \text{Int}(F)$. $\phi^{-1}(k)$ is an isotropic submanifold of X , and so a closed horizontal lift, Y_k , will exist if the holonomies of generators of the fundamental group of $\phi^{-1}(k)$ are trivial. By the properties of Delzant polytopes, we can represent generators of $\pi_1(\phi^{-1}(k))$ by orbits of one-parameter subgroups $\exp(tA)$ of period $T = 2\pi$, and with $A \in \mathfrak{k}$ integral.

Fix $p \in P$ above $\phi^{-1}(k)$, and note that the curve $\exp(tA) \cdot p$ is 2π periodic (we are denoting the action of K on P by a dot). Therefore

$$p = \exp(2\pi A) \cdot p = \exp(2\pi \tilde{\xi}_A) \circ \exp(2\pi H_A(p) \partial_\theta)(p).$$

Since A and k are integral, $H_A(p) = \langle A, k \rangle \in \mathbb{Z}$, and therefore $\exp(2\pi H_A(p) \partial_\theta)(p) = p$. Therefore

$$p = \exp(2\pi \tilde{\xi}_A)(p),$$

that is, the holonomy of an orbit of $\exp(tA)$ in $\phi^{-1}(k)$ is trivial. This proves (1).

To prove (2), note that $\forall A \in \mathfrak{k}$ the section s_k satisfies

$$\mathcal{L}_{\xi_A^\#}(s_k^\flat) = \sqrt{-1} \langle A, k \rangle s_k^\flat.$$

Taking into account that $\mathcal{L}_{\partial_\theta} s_k^\flat = \sqrt{-1} s_k^\flat$, we obtain using (7.1) that $\mathcal{L}_{\tilde{\xi}_A}(s_k^\flat) = 0$ at points over $\phi^{-1}(k)$. Since this is true $\forall A \in \mathfrak{k}$, s_k^\flat is constant on Y_k . It is not zero because, as we have seen, $\langle s_k, s_k \rangle$ is in fact maximal on $\phi^{-1}(k)$. \square

The main result of this section is:

Proposition 7.2. *Let $k \in [\Delta]$ and $Y_k \subset P$ as in the previous proposition. Let ν be the lift to Y_k of a K -invariant density on $\phi^{-1}(k)$. Then the projection, $\Pi_1(\delta_{Y_k})$, on \mathcal{H}_1 of the resulting delta function on Y_k is a non-zero constant times s_k .*

Proof. Let $t_k = \Pi_1(\delta_{Y_k})$. We begin by clarifying that, as a distribution, t_k is defined by the identity

$$(7.2) \quad (t_k, u) = (\delta_{Y_k}, \overline{\Pi_1(\overline{u})}),$$

for u a test function on P . Therefore, if $u \in \mathcal{H}_1$,

$$(7.3) \quad \langle t_k, u \rangle_{L^2} = (t_k, \overline{u}) = \int_{Y_k} \overline{u} \nu.$$

For any $A \in \mathfrak{k}$ let us now compute $\mathcal{L}_{\xi_A^\#}(t_k)$. Using that $[\mathcal{L}_{\xi_A^\#}, \Pi_1] = 0$, if u is a test function on P

$$(\mathcal{L}_{\xi_A^\#}(t_k), u) = - \int_{Y_k} \mathcal{L}_{\xi_A^\#}(u_1) \nu$$

where $u_1 = \overline{\Pi_1(\overline{u})}$. By (7.1) this equals

$$- \int_{Y_k} \mathcal{L}_{\tilde{\xi}_A}(u_1) \nu + \sqrt{-1} \langle A, k \rangle \int_{Y_k} u_1 \nu.$$

The first integral is zero, because Y_k is horizontal and $\mathcal{L}_{\xi_A^\#} \nu = 0$. The second term is

$$\sqrt{-1} \langle A, k \rangle \int_{Y_k} \overline{\Pi_1(\overline{u})} \nu = \sqrt{-1} \langle A, k \rangle (t_k, u),$$

using (7.2). Therefore $\mathcal{L}_{\xi_A^\sharp}(t_k) = \sqrt{-1} \langle A, k \rangle t_k$, that is, t_k satisfies the same ODEs as s_k^\flat , and so necessarily $t_k = C_k s_k^\flat$ for some constant C_k . To show that this constant is not zero note that

$$C_k = \langle t_k, s_k^\flat \rangle_{L^2} = \int_{Y_k} \overline{(s_k)^\flat} d\nu \neq 0$$

by (7.3) and part (2) of the previous lemma (in fact C_k is equal to the volume of Y_k times the conjugate of the constant value of s_k^\flat on Y_k). \square

It is natural to ask if the analogue of the previous proposition holds for other integrable systems on Kähler manifolds, for example the Gelfand-Cetlin system. We hope to return to this problem.

REFERENCES

- [1] D. Borthwick, T. Paul and A. Uribe, “Legendrian distributions with applications to relative Poincaré series”, *Invent. Math.* **122** (1995), 359–402.
- [2] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Math. Studies 99, Princeton U. Press, Princeton, NJ, 1981.
- [3] D. Burns and V. Guillemin, “Potential functions and actions of tori on Kähler manifolds”, *Comm. Anal. Geom.* **12** (2004) no. 1-2, 281–303.
- [4] D. Calderbank, L. David and P. Gauduchon, “About the Guillemin formula for the Kähler potential of a toric manifold”, *J. Symplectic Geom.* **1** (2003), no. 4, 767–784.
- [5] T. Delzant, “Hamiltoniens periodiques et image convexe de l’application moment”, *Bull. Soc. Math. France*, **116** (1988), 315–339.
- [6] V. Guillemin, “Kähler structures on toric varieties”, *JDG* **40** (1994), 285–309.
- [7] V. Guillemin and S. Sternberg, “Riemann sums over polytopes”, *Ann. Inst. Fourier* (to appear).
- [8] B. Shiffman, T. Tate and S. Zelditch, “Distribution laws for integrable eigenfunctions”, *Ann. Inst. Fourier* (Grenoble) **54** (2004) no. 5, 1497–1546.
- [9] A. N. Tyurin, “On Bohr-Sommerfeld bases” (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), no. 5, 163–196; translation in *Izv. Math.* **64** (2000), no. 5, 1033–1064.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109
E-mail address: dburns@umich.edu

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139
E-mail address: vwg@mit.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109
E-mail address: uribe@umich.edu